

“Quantumness” versus “Classicality” of Quantum States

Berry Groisman¹, Dan Kenigsberg² and Tal Mor²

1. Centre for Quantum Computation, DAMTP, Centre for Mathematical Sciences,
University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UNITED KINGDOM

2. Computer Science Department, Technion, Haifa 32000 ISRAEL

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Entanglement is one of the pillars of quantum mechanics and quantum information processing, and as a result the *quantumness* of nonentangled states has typically been overlooked and unrecognized. We give a robust definition for the classicality versus quantumness of a single multipartite quantum state, a set of states, and a protocol using quantum states. We show a variety of nonentangled (separable) states that exhibit interesting quantum properties, and we explore the “zoo” of separable states; several interesting subclasses are defined based on their diagonalizing bases, and their non-classical behavior is investigated.

Introduction: Consider an isolated discrete classical system with N distinguishable states. The most general state of the classical system is a probabilistic distribution over these distinguishable states. Now consider its counterpart, an isolated discrete *quantum* system. Its most general state is a probabilistic mixture of pure states drawn from an N -dimensional Hilbert space. Yet, in various special cases, the quantum state seems to be identical to a classical probability distribution. Similarly, in various special cases, a quantum protocol using a set of quantum states seems to be practically identical to a classical protocol which is using a classical set of states. Our first goal is to define such special quantum states that are equivalent to classical probability distributions; we also define sets of classical states and classical protocols.

Quantumness of states (for instance, their “quantum correlations”) is often associated with their entanglement, and it is sometimes even assumed (explicitly or implicitly) that non-entangled states can be considered “classical”. We argue that this is not the case, because some (actually, most) non-entangled states do exhibit non-classical features. Intuitively speaking, only quantum states that correspond *exactly* to a classical probability distribution can potentially be considered classical; most nonentangled states can only be written as a probability distribution over tensor-product quantum states, e.g., for bipartite systems $\rho_{\text{sep}} = \sum_i p_i |\phi_i\rangle_A |\psi_i\rangle_B \langle\phi_i|_A \langle\psi_i|_B$, hence do not usually resemble any conventional distribution over classical states. While entanglement is extensively analyzed and quantified (see [1, 2], and references therein), the “quantumness” of nonentangled (separable) states has typically been overlooked and unrecognized. Our second goal is to present the quantumness exhibited by various separable states, and to explore the “zoo of separable states”. Our last goal is to define (and make use of) measures of quantumness $\mathcal{Q}(\rho)$ that vanish on any classical state $\rho_{\text{classical}}$.

Classicality of Quantum States and Quantum Protocols: If a quantum state or a quantum protocol has an exact classical equivalent it cannot present any interest-

ing nonclassical properties nor any advantage over its analogous classical counterpart. The state(s) of the quantum system can then potentially be considered “classical”. For instance, if a single quantum system is prepared in one of the states $|0\rangle, |1\rangle, |2\rangle$, etc., and is then measured in this computational basis, there is nothing genuinely quantum in that process. Tensor product states of multipartite system can also be considered classical. Consider a set of states in the computational basis, e.g., $\{|00\rangle; |01\rangle; |10\rangle; |11\rangle\}$; this set has a strict classical analogue — the classical states $\{00; 01; 10; 11\}$. As long as no other quantum states are added to the set (or appear in a protocol which is using these states), the analogy is kept, so these quantum states can be considered classical. Tensor product states such as $|-\rangle|0\rangle|+\rangle$ (where $|\pm\rangle = [|0\rangle \pm |1\rangle] / \sqrt{2}$) can also be considered classical as we soon explain.

First, we define classical bases. We justify our claim that any such basis presents no quantumness, and we justify (via many examples) why bases that do not follow our “classicality” definition are “quantum”.

We start with a single system and then move to bipartite and multipartite systems:

Definition 1 *Let A be a quantum system. Any orthonormal basis $\{|i\rangle_A\}$ of A can be considered as a classical basis of the system.*

For example, the computational basis $\{|0\rangle; |1\rangle\}$ of a single qubit is obviously classical. The Hadamard basis $\{|+\rangle; |-\rangle\}$ is also classical.

One may argue that our definition is too flexible and that Nature allows only one basis to be classical. For instance an alternative for Def. 1 is

Let A be a quantum system with a single *preferred* orthonormal basis $\{|i\rangle_A\}$, in the sense that measurements can only be performed in this basis. Only this basis can be considered as a *classical basis* of the system.

We do not agree to that narrower definition. First, nothing in conventional quantum theory favors one of the sys-

tem's bases over any other. Second, although in the more general *relativistic quantum field theory* it is commonly believed that Nature generally provides a preferred basis, on time-scales sufficiently short (e.g., short enough for performing quantum computation), all bases are equivalent.

We now move to defining classical bases for bipartite and multipartite systems.

Definition 2 Let A and B be two quantum systems with orthonormal bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ respectively. The tensor-product basis $\{|i\rangle_A \otimes |j\rangle_B\}_{ij}$ is a classical basis of the bipartite system.

Definition 3 (recursive) Let A be a (bipartite or multipartite) quantum system with a classical basis $\{|i\rangle_A\}$, and let B be a unipartite quantum system with an orthonormal basis $\{|j\rangle_B\}$. The tensor-product basis $\{|i\rangle_A \otimes |j\rangle_B\}_{ij}$ is a classical basis of the composite AB system.

The redundancy in Defs. 2–3 is kept for readability.

Let us see a few examples. For two qubits, the computational basis is classical, as well as the basis $\{|++\rangle; |+-\rangle; |-+\rangle; |--\rangle\}$. On the other hand, the Bell basis $\{|\Phi_{\pm}\rangle; |\Psi_{\pm}\rangle\}$ is obviously non-classical, and more interestingly, even the basis $\{|00\rangle; |01\rangle; |1+\rangle; |1-\rangle\}$ is non-classical, too.

Having identified classical bases, we proceed to define a classical state and a set of classical states.

Definition 4 A state ρ is a classical state, iff there exists a classical basis $\{|v_i\rangle\}$ in which ρ is diagonal.

Following our definition, any (single) state ρ (either pure or mixed) of a single system S can always be considered classical. A joint state of two or more quantum systems can also either be pure or mixed. If it is pure it is either a tensor product state or an entangled state. Following the classicality definitions, any such tensor-product state is classical while any such entangled state is nonclassical. For mixed bipartite or multipartite states the situation is much more complicated: Tensor-product mixed states are obviously still classical as each subsystem can be diagonalized in a classical basis of its own. Entangled mixed states are obviously nonclassical. Between these two extremes we can find a zoo of separable—yet quantum—states.

We made this definition independently of a similar definition due to Ref. [3, see Sec. 5]; they use the name “(properly) classically correlated states” which is more precise, yet longer, than our term “classical states”.

Prior to dealing with separable quantum states we provide two additional useful definitions.

Definition 5 A set of states $\rho_1 \dots \rho_k$ is a classical set iff all ρ_i are diagonalizable in a single classical basis.

If a quantum protocol (be it computational, cryptographic, or any other physical process) is limited to a classical set of states, the process has an exact classical equivalent, and cannot present any advantage over an analogous classical protocol. More formally:

Definition 6 A protocol (in quantum information processing) is classical iff all states involved in it belong to a single classical set of states.

If a protocol involves two or more pure nonorthogonal states it cannot be considered classical [see [4] for a thorough analysis of the quantumness of protocols involving only pure states.] Yet following our definitions, even protocols involving only pure orthogonal product-states might be highly quantum; and similarly, even a single bipartite mixed separable state can be highly nonclassical.

Nonclassicality of Separable States: Let us prove the quantumness of several interesting separable states.

1.— Pseudo-pure states.

A state of the form $\epsilon|\psi\rangle\langle\psi| + \frac{1-\epsilon}{N}I$ is called a pseudo-pure state (PPS) as the part with the coefficient ϵ transforms as if the state was a pure state. PPSs focus wide interest based on theoretical and experimental grounds. It has been shown [5] that there is a volume of *separable* PPSs around the totally-mixed state I/N ; every PPS with low-enough ϵ is separable. This fact was even used to argue that experiments which produce such low- ϵ states are not truly quantum. It was later argued, however, that albeit being separable, these states do exhibit non-classical effects [6]. Using our definitions we see that:

Proposition 7 A PPS $\rho_{\epsilon} = \epsilon\rho + \frac{1-\epsilon}{N}I$ is quantum iff ρ is, for any $\epsilon > 0$.

Proof Any diagonalizing basis of ρ_{ϵ} also diagonalizes ρ , independently of ϵ . Since ρ is quantum, it is not diagonalizable in a classical basis, and so is ρ_{ϵ} . \square

This is true for any system dimension. As a special case for $N = 4$, a separable Werner state [7] $\chi = \epsilon|\Psi_{-}\rangle\langle\Psi_{-}| + \frac{1-\epsilon}{4}I$ is nonclassical for any $0 < \epsilon \leq \frac{1}{3}$ (see also [8] for a different demonstration of nonclassicality of the Werner states). Note that the Werner state is also separable and nonclassical for any $-\frac{1}{3} \leq \epsilon < 0$.

2.— States used for quantum key distribution.

The original quantum key distribution protocol, the BB84 protocol, involves qubits of four different states: $|0\rangle$, $|1\rangle$, $|+\rangle$, and $|-\rangle$, sent from Alice to Bob. The protocol may also be described in a less conventional manner [9], where Alice sends in two steps either the state $\rho_{0(\text{BB84})} = \frac{1}{2}[|00\rangle\langle 00| + |1+\rangle\langle 1+|]$ to represent ‘0’ or $\rho_{1(\text{BB84})} = \frac{1}{2}[|01\rangle\langle 01| + |1-\rangle\langle 1-|]$ to represent ‘1’; the right-hand-qubit is sent first and the left-hand-qubit is sent later on in order to reveal the basis of the first qubit.

Proposition 8 $\rho_{0(\text{BB84})}$ is not classical; so is $\rho_{1(\text{BB84})}$.

Proof Any diagonalizing product basis of $\rho_{0(\text{BB84})}$ includes $|0\rangle_A \otimes |0\rangle_B$ and $|1\rangle_A \otimes |+\rangle_B$. That basis cannot be classical, as Bob's parts, $|0\rangle_B$ and $|+\rangle_B$, are not orthogonal and hence cannot be members of a single classical basis. The same reasoning applies to $\rho_{1(\text{BB84})}$, too. \square

Thus, although all the four states involved in the protocol $|00\rangle$, $|1+\rangle$, etc. are mutually orthogonal tensor-product states, the protocol is highly “quantum”.

3.— States that present nonlocality without entanglement.

Various sets of states proposed in [10, 11] define processes that exhibit nonlocal quantum behavior although none of the participating states is entangled. In particular, spatially separated parties cannot reliably distinguish between different members of the set (albeit comprising of mutually orthogonal direct product states!) without assistance of entanglement. For instance, the set $\{|01+\rangle; |1+0\rangle; |+\!01\rangle; |---\rangle\}$ is nonclassical.

4.— The Bernstein-Vazirani Algorithm.

The Bernstein-Vazirani algorithm [12] generates no entanglement (see [13]). However, it is clearly a quantum algorithm, with no classical equivalent. It makes use of states from the computational *and* Hadamard bases, which are not simultaneously diagonalizable in a single classical basis.

A Zoo of Separable States: Within the set of all separable states we identify some interesting subsets based on their diagonalizing bases.

First let us consider the classical states *Class*, the states diagonalized in a classical basis: A bipartite state (this argument easily extends to multipartite states) is classical if, and only if, Alice and Bob can perform a measurement in its (classical) diagonalizing basis via local orthogonal measurements, without exchanging any message (classical or quantum) and without disturbing the state.

The notion of diagonalizing basis is now used to define more subsets of the separable states. Ref. [11] defines a *complete product basis* (CPB) as follows: A CPB is a complete orthonormal basis of a multipartite Hilbert space, where each basis element is a (tensor) product state. We define the set of CPB-states as follows:

Definition 9 A state ρ is a CPB-state iff it is diagonalizable in a CPB.

Clearly, all classical states are CPB states; but not vice versa. Thus, in a multipartite finite-dimensional Hilbert space $\text{Class} \subset \text{CPB} \subset \text{SEP} \subset \mathcal{H}_{\text{total}}$. For example, $\rho_{0(\text{BB84})}$ and $\rho_{1(\text{BB84})}$ are nonclassical CPB-states diagonalized in the CPB $\{\rho_{00}; \rho_{01}; \rho_{1+}; \rho_{1-}\}$. Note that local operations and unidirectional classical communication, but *without* adding the ability to “forget”, are sufficient for converting the BB84 states into classical states. These operations are a very special case of

the well-known LOCC (local operations and classical communication) that *include the ability to forget*, and that are therefore sufficient for generating any separable state. A slightly more complicated (qubit plus qutrit) state, $\rho = \frac{1}{3} [|00\rangle\langle 00| + |1+\rangle\langle 1+| + |+\!2\rangle\langle +2|]$ requires local operations (again, without “forgetting”) and bidirectional classical communication in order for it to be converted into a classical state. We call these two types of CPB-states “unidirectional CPB-states” and “multi-directional CPB-states” respectively.

Interestingly, there are CPB-states that belong to neither subsets: consider a state built from a probability distribution over *all* the eight states $[10] \{|01\pm\rangle; |1\pm 0\rangle; |\pm 01\rangle; |000\rangle; |111\rangle\}$; although it is a CPB-state, such a state cannot be converted into a classical states unless quantum communication is allowed, or unless the general LOCC (including the power of “forgetting”) are allowed. Thus, we specify also a third subset of the CPB states — “Q-convertible CPB-states”.

Let V be an orthonormal basis of a subspace of a multipartite Hilbert space \mathcal{H} , where each basis element is a (tensor) product state. Ref. [11] defines that V is an *unextendible product basis* (UPB) if the subspace $\mathcal{H} - \text{span}\{V\}$ contains no product state. We define the set of UPB-states as follows:

Definition 10 A separable state ρ is a UPB-state iff it is diagonalizable in a UPB.

Note that a UPB-state [11] such as $\rho_{\epsilon \text{UPB}} = (1 - 6\epsilon)\rho_{01-} + \epsilon\rho_{1-0} + 2\epsilon\rho_{-01} + 3\epsilon\rho_{---}$, proves that there are UPB-states that are not in CPB. Note also that with $\epsilon \rightarrow 0$, this state is infinitesimally close to a classical state. More relations and borderlines between these sets and also the set EB (see below) will be explored in future research [14].

We identified another class of UPB-states that can be proven to be non-classical:

Proposition 11 The uniform mixture of UPB elements $\rho_{\text{UPB}} = (\rho_{01+} + \rho_{1+0} + \rho_{+01} + \rho_{---})/4$ is nonclassical.

Proof Assume that ρ_{UPB} is classical. The same classical basis that diagonalizes it, also diagonalizes the state $I/4 - \rho_{\text{UPB}}$. However, this contradicts the fact that it is *bound-entangled* [11] and therefore quantum. \square

The last set we define is the set EB of states diagonalized only in a non-product basis. As we had already seen, many separable states belong to this EB set, e.g., various PPS and Werner states. Obviously, all non-separable states also belong to this set.

Measures of quantumness: A measure of nonclassicality (quantumness), $\mathcal{Q}(\rho)$, of a state ρ has to satisfy two conditions; (a) $\mathcal{Q}(\rho) = 0$ iff ρ is classical, (b) $\mathcal{Q}(\rho)$ is invariant under local unitary operations. One might also expect a third condition; (c) $\mathcal{Q}(\rho)$ is monotonic under local operations (without classical communication)

[15]; Yet, condition (c) is not always satisfied by quantum states: The classical state $\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|03\rangle\langle 03|$ of a 2×4 system can be converted to $\rho_{0(\text{BB84})}$ just by the power of forgetting—Bob redefines his qu-quadrant as two qubits with $|0\rangle_{\text{quad}} = |00\rangle$ and $|3\rangle_{\text{quad}} = |1+\rangle$, and forgets his first qubit.

A class of measures of quantumness of ρ is defined as

$$\mathcal{Q}_D(\rho) = \min_{\rho_c} D(\rho, \rho_c) \quad (1)$$

where D is any measure of distance between two states such that the conditions (a)-(b) are satisfied, and the minimum is taken over all classical states ρ_c . One of the natural candidates for D is the relative entropy $S(\rho||\rho_c) = \text{tr } \rho \log \rho - \text{tr } \rho \log \rho_c$, in which case we refer to it as $\mathcal{Q}_{\text{rel}}(\rho)$ — the *relative entropy of quantumness*. The benefit of using the relative entropy as a measure is that it was extensively studied for measuring entanglement [1] (relative to the closest separable state). Thus, we can adopt and make use of some known results, and we can also monitor the connection between the quantumness of states and their entanglement. Other measures (or their variants) that can potentially be very useful are the *fidelity of quantumness* and *Von Neumann mutual information* that will be explored in future research [14].

For bipartite pure states, the relative entropy of quantumness equals its entropy of entanglement. In other words, a pure state is quantum as much as it is entangled. Any bipartite entangled state $|\Psi\rangle$ can be written in a Schmidt decomposition $|\Psi\rangle = \sum_{i=1}^d c_i |ii\rangle_{AB}$, where $c_i \geq 0$ and $d = \min[d_A, d_B]$, d_A, d_B are dimensions of local Hilbert spaces. If we use the relative entropy of entanglement then the closest separable state [1] is

$$\sigma_{cl} = \sum_{i=1}^d (c_i)^2 |ii\rangle\langle ii|_{AB}. \quad (2)$$

This state happens to be also classical, and thus the relative entropy of quantumness (which is equal to its relative entropy of entanglement) is $\mathcal{Q}_{\text{rel}}(\Psi) = -\sum_i (c_i)^2 \log[(c_i)^2]$. [The classical state σ_{cl} lies on entangled-separable boundary.] Note that the quantumness of a maximally entangled state is $\mathcal{Q}_{\text{rel}}(\Psi_{ME}) = \log d$.

Let us present some mixed states for which their quantumness can easily be calculated: According to [1, Th. 4], σ_{cl} is the separable state that minimizes $S(\rho_p||\sigma_{cl})$ for any state of the form $\rho_p = p |\Psi\rangle\langle\Psi| + (1-p)\sigma_{cl}$, too. Therefore, the relative entropy of entanglement of ρ_p equals to its relative entropy of quantumness.

Given any bipartite state ρ_{AB} , let its *Schmidt basis* be the (classical) basis diagonalizing $\text{tr}_B \rho_{AB} \otimes \text{tr}_A \rho_{AB}$. Let ρ_{Sch} be produced from ρ_{AB} by writing it in its Schmidt basis and having all off-diagonal elements zeroed. The state ρ_{AB} and its Schmidt state yield identical classical correlations if measured in the Schmidt basis.

The Schmidt state can be found very useful for defining quantumness for any state ρ_{AB} , as ρ_c is usually unknown; instead of using Eq. (1) as a measure, one can directly refer to the distance between a state ρ_{AB} and its corresponding Schmidt state:

$$\mathcal{Q}_D(\rho) = D(\rho_{AB}, \rho_{\text{Sch}}) \quad (3)$$

as a measure of quantumness of a state. If we now use the relative entropy, the resulting measure satisfies conditions (a) and (b).

We saw above, that for a pure bipartite state the Schmidt state $\rho_{\text{Sch}} = \sigma_{cl}$ is the closest classical state. One might conjecture that for any bipartite state ρ , the closest classical state (using relative entropy measure) is its Schmidt-state ρ_{Sch} . This however, is not true. For instance, we checked the CPB-state $\rho_{0(\text{BB84})}$ which is useful in quantum key distribution; it is interesting to note that either the classical state $\frac{1}{2}\rho_{00} + \frac{1}{4}\rho_{10} + \frac{1}{4}\rho_{11}$ or the classical state $\frac{1}{4}\rho_{0+} + \frac{1}{4}\rho_{0-} + \frac{1}{2}\rho_{1+}$, are actually closer to $\rho_{0(\text{BB84})}$ than its Schmidt state — a state diagonal in the classical basis (known as the Breidbart basis) $\{\rho_{0b_0}; \rho_{0b_1}; \rho_{1b_0}; \rho_{1b_1}\}$ (where $|b_0\rangle = \cos \frac{\pi}{8}|0\rangle - \sin \frac{\pi}{8}|1\rangle, |b_1\rangle = \sin \frac{\pi}{8}|0\rangle + \cos \frac{\pi}{8}|1\rangle$). It is easy to verify numerically that the above two states are the closest ones to $\rho_{0(\text{BB84})}$, hence can be used for calculating its relative entropy of quantumness. We also proved this fact analytically but the proof is too long and therefore is not included in this Letter, but is left for an extended paper [14]. The entropy of quantumness relative to the Schmidt state is different in this case of course.

Summary: In this Letter we gave definitions for classical states and protocols in quantum information processing. We explored the “zoo” of separable states, we gave a good number of examples and we defined some useful measures for the quantumness of non-classical states. Our measures and our analysis are mainly based on the notions of “diagonalizing basis” and the “Schmidt basis” (which are identical in the case of pure entangled states). Other measures of quantumness have been defined and used previously: Ref. [8] defines the *quantum discord* between the parts of a bipartite state. Ref. [3] extensively uses the *quantum information deficit* measure of quantumness, and the relative entropy of quantumness (which we use independently). Section 5 in [3] provides very interesting subclasses — yet, different from ours — of the separable states. Their class of “informationally nonlocal” states seems to be identical to our two subclasses — the UPB-states and the unconvertible CPB-states.

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